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# Force Method Revisited

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**Computational mechanics is an important field in engineering with a strong foundation in continuum mechanics and finite element analysis. The standard and integrated force methods are reviewed, and a structural analysis technique using the finite element force method based on the complementary strain energy is proposed. Similar to the standard force and the integrated force methods, the force method based on the complementary strain energy uses the same equilibrium equations. However, the compatibility conditions are satisfied through the complementary strain energy. The Gauss elimination technique has been employed successfully to generate automatically a basis determinate structure and redundant members. Extension of the force method to dynamics and benchmark problems with displacement–stress and frequency constraints are presented.**

## I. Introduction

THE concept of equilibrium of forces and compatibility of deformations is fundamental to analysis methods for solving problems in structural mechanics. The equilibrium equations need to be augmented by the compatibility conditions because the equilibrium equations are indeterminate by nature, and determinacy is achieved by adding the compatibility conditions. Generally, two analytical methods (displacement and force) are available to analyze determinate and indeterminate structures.

Structural analysis and optimization algorithms developed in recent years have generally been based on the displacement method.[1–6] Commercial finite element programs are based on the displacement method, and very few investigations have been reported in structural optimization using the finite element force method.[7–9] The displacement method is an efficient approach; however, for stress–displacement constraints, it loses its advantages for size optimization when the number of stress constraints is larger than the number of displacement constraints and for topology optimization when the structural strength is the primary design concern. Additionally, when the structure is not highly redundant, that is, the number of redundant elements is lower than the displacement degrees of freedom, analysis through the force method is more efficient than the displacement method.

In the standard (or classical) form of the force method (SFM), it is very difficult to generate the compatibility conditions. Splitting the given structure into a determinate basis structure and redundant members generates the compatibility in the classical force method. The compatibility conditions are written by establishing the continuity of deformations between redundant members and the basis structure.[10] Several schemes have been devised to generate automatically redundant members and the basis determinate structure,[11–13] however, with limited success. In the integrated force method (IFM) developed by Patnaik,[14,15] Patnaik and Nagaraj,[16] Vijayakumar et al.,[17] and Patnaik et al.,[18] both equilibrium equations and compatibility conditions are solved simultaneously. The generation of compatibility equations is based on extending St. Venant's theory of elasticity strain formulation to discrete structural mechanics and eliminating the displacements in the deformation–displacement relation.

In this paper, the general formulations of the standard force and integrated force methods are reviewed. An alternative method to generate directly the compatibility matrix in the integrated force method is proposed. Next, the development of the formulation for the force method based on the complementary strain energy is introduced. The advantages and limitations of the three methods are discussed, and a comparison of the force and displacement methods is carried out. Last, the application of the force method in frequency and dynamic problems is studied.

## II. Force Method Formulations

The force method of analysis is based on the equations of equilibrium expressed in terms of the element forces. For particular and simple structures, these equations are sufficient to determine all of the forces and subsequently the element stresses and displacements. Such structures are said to be statically determinate. However, for general and complex structures, the number of element forces exceeds the number of available equations of equilibrium, and the structure is said to be statically indeterminate (or redundant). For such cases, the equations of equilibrium are insufficient to obtain solutions for the element forces, and therefore, additional equations are required. These additional equations are in the form of compatibility conditions on displacements.

### A. SFM

The standard (classical) force method is basically based on the virtual force method. According to this method, virtual complementary work is obtained by multiplying virtual forces by real displacements. The principle of virtual complementary work states that

$$\delta W^* = \delta \Pi^* \tag{1}$$

where $W^*$ and $\Pi^*$ are the complementary work and complementary strain energy, respectively.

Let us assume now that a virtual load $\delta P$ in the direction of the displacement $U$ is applied in a discrete or discretized structure $(n, m)$, where structure denotes a type of structure (truss, frame, plate, shell, or a combination) under the action of a system of forces that induce internal element forces $F$ and $n$ and $m$ are the force degrees of freedom (FOF) and displacement degrees of freedom (DOF), respectively. The virtual complementary work is then

$$\delta W^* = U \delta P \tag{2}$$

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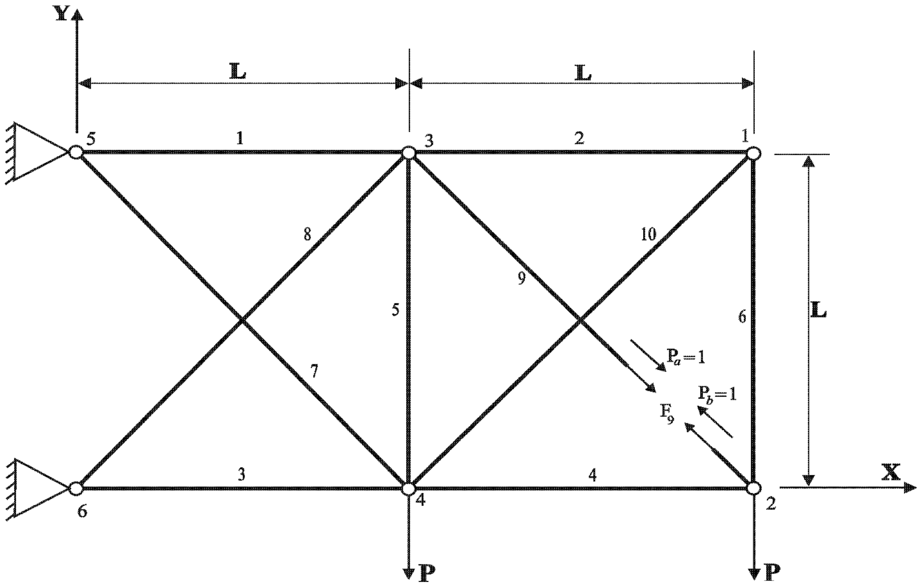


**Fig. 1 Redundant 10-bar truss structure.**

Now, because of the virtual force $\delta P$, the virtual complementary strain energy on the $i$th element is given by

$$\delta\Pi^{*(i)} = \delta F^{(i)^T}\Delta^{(i)} \tag{3}$$

and for the complete structure

$$\delta\Pi^{*} = \delta \boldsymbol{F}^T \boldsymbol{\Delta} \tag{4}$$

where $\Delta$ is the deformation vector and

$$\delta \boldsymbol{F} = \begin{bmatrix} \delta F^{(1)} \\ \delta F^{(2)} \\ \vdots \\ \delta F^{(i)} \\ \vdots \end{bmatrix}, \qquad \boldsymbol{\Delta} = \begin{bmatrix} \Delta^{(1)} \\ \Delta^{(2)} \\ \vdots \\ \Delta^{(i)} \\ \vdots \end{bmatrix} \tag{5}$$

For linear elasticity, $\delta\boldsymbol{F}$ in Eq. (5) can be expressed as

$$\delta \boldsymbol{F} = \begin{bmatrix} \hat{F}^{(1)} \\ \hat{F}^{(2)} \\ \vdots \\ \hat{F}^{(i)} \\ \vdots \end{bmatrix} \delta P = \hat{\boldsymbol{F}}\delta P \tag{6}$$

where $\hat{\boldsymbol{F}}$ is the vector of element forces due to $\delta P = 1$. Now, substituting Eq. (6) into Eq. (4) and subsequently Eqs. (4) and (2) into Eq. (1), we obtain

$$U\delta P = \hat{\boldsymbol{F}}^T\boldsymbol{\Delta}\delta P \qquad \text{or} \qquad U = \hat{\boldsymbol{F}}^T\boldsymbol{\Delta} \tag{7}$$

The deformation vector $\boldsymbol{\Delta}$ may be related to the element forces using the relation

$$\boldsymbol{\Delta} = \boldsymbol{G}\boldsymbol{F} \tag{8}$$

where $\boldsymbol{G}$ is the $(n \times n)$ flexibility matrix of the structure. Substituting Eq. (8) into Eq. (7), we obtain

$$U = \hat{\boldsymbol{F}}^T\boldsymbol{G}\boldsymbol{F} \tag{9}$$

Equation (9) represents the matrix form of the unit-load theorem[10] for a single displacement $U$.

The unit-load theorem just introduced has been applied to external forces. However, it may be generalized to be applicable to internal forces as well. In this case, the resulting displacements represent internal relative displacements that must be equal to zero to satisfy continuity of deformations (compatibility conditions). To illustrate this point, consider a two-dimensional redundant truss structure subject to external loads $P$ at nodes 2 and 4, as shown in Fig. 1. The relative displacement $\boldsymbol{\Delta}$, on the individual elements due to $P$ are given by Eq. (8). Let us assume that a fictitious cut is introduced in the diagonal member 9 near joint 2. The force $F_9$ (which existed in the member before the cut was introduced) must be supplied by some external means to maintain equilibrium with the external loading. Note that this particular structure is redundant with degree of redundancy $r = n - m = 10 - 8 = 2$. When introducing the cut in the diagonal member 9, it is reduced to the degree of redundancy $r = n - m = 9 - 8 = 1$.

If a unit load $P_a$ is applied at the cut in the direction $3 \Rightarrow 2$ (from node 3 to node 2), as shown in Fig. 1, the unit load theorem gives a deflection $U_a$, obtained using Eq. (9) as

$$U_b = \hat{\boldsymbol{F}}_b^T\boldsymbol{G}\boldsymbol{F} \tag{10}$$

where $\hat{\boldsymbol{F}}_a$ is the vector of statically equivalent element forces due to $P_a = 1$. If, instead of $P_a$, a unit force $P_b$ is applied in the direction $2 \Rightarrow 3$, again from Eq. (9) it follows that

$$U_b = \hat{\boldsymbol{F}}_b^T\boldsymbol{G}\boldsymbol{F} \tag{11}$$

where $\hat{\boldsymbol{F}}_b$ is the vector of statically equivalent element forces for $P_b = 1$. To preserve the continuity of deformations at the point of the cut, we must have

$$U_a = -U_b \qquad \text{or} \qquad U_a + U_b = 0 \tag{12}$$

Substituting Eqs. (10) and (11) into Eq. (12), we may have

$$(\hat{\boldsymbol{F}}_a + \hat{\boldsymbol{F}}_b)^T\boldsymbol{G}\boldsymbol{F} = \boldsymbol{0} \qquad \text{or} \qquad \hat{\boldsymbol{F}}_C^T\boldsymbol{G}\boldsymbol{F} = \boldsymbol{0} \tag{13}$$

where $\hat{\boldsymbol{F}}_C = \hat{\boldsymbol{F}}_a + \hat{\boldsymbol{F}}_b$. The matrix $\hat{\boldsymbol{F}}_C$ can be interpreted as an internal-force system representing statically equivalent forces due to a unit force applied across the fictitious cut.

A number of cuts equal to the degree of redundancy can be introduced until the structure is reduced to a statically determinate system. For a structure with $r = n - m$ degree of redundancy, $r$ cuts are required. For each cut, according to Eq. (13), the following set of equations is obtained:

$$\begin{aligned}\hat{F}_{C1}^T G F &= 0\\ \hat{F}_{C2}^T G F &= 0\\ &\cdots\cdots\cdots\\ \hat{F}_{Cr}^T G F &= 0\end{aligned} \tag{14}$$

The preceding equations are $r$ compatibility equations, and these can be combined into a single matrix equation:

$$CGF = 0 \tag{15}$$

where $C = [\hat{F}_{C1} \; \hat{F}_{C2} \; \cdots \; \hat{F}_{Cr}]^T$ is the $(r \times n)$ compatibility matrix.

For derivation of the equilibrium equations, the components of the element forces in the directions of all of the DOF at the nodes of the discrete or discretized structure are algebraically summed and subsequently equated to the corresponding components of the externally applied loads. The $m$ equilibrium equations can be combined into a single matrix equation as

$$QF = P \tag{16}$$

where $Q$ is the $(m \times n)$ equilibrium matrix whose coefficients are the direction cosines used in resolving the element forces $F$ and $P$ is a vector of external forces applied in the direction of the DOF. If the structure is statically determinate, that is, $m = n$ and the rank of the matrix $Q = m$, the element forces $F$ can be calculated directly from Eq. (16). For this special case,

$$F = Q^{-1} P \tag{17}$$

For statically indeterminate structures, that is, $m < n$ and the rank of the matrix $Q = m$, the equations of equilibrium are not sufficient to determine the forces $F$. The required additional equations are supplied by the $r$ compatibility conditions given by Eq. (15).

The element forces $F$ may be partitioned symbolically into statically determinate forces $F_d$ and redundant forces $F_r$, that is,

$$F = \begin{bmatrix} F_d \\ F_r \end{bmatrix} \tag{18}$$

Also, the equilibrium matrix $Q$ may be consistently partitioned for $F_d$ and $F_r$ as

$$Q = \begin{bmatrix} Q_d & Q_r \end{bmatrix} \tag{19}$$

Thus, one can rewrite the equilibrium equation (16) as

$$[Q_d \quad Q_r] \begin{bmatrix} F_d \\ F_r \end{bmatrix} = P \tag{20}$$

When $F_d$ is solved for,

$$F_d = \begin{bmatrix} Q_d^{-1} & -Q_d^{-1} Q_r^{-1} \end{bmatrix} \begin{bmatrix} P \\ F_r \end{bmatrix} \tag{21}$$

Combining $F_d$ from Eq. (21) and $F_r$, we obtain

$$F = \begin{bmatrix} F_d \\ F_r \end{bmatrix} = \begin{bmatrix} Q_d^{-1} & -Q_d^{-1} Q_r \\ 0 & I_r \end{bmatrix} \begin{bmatrix} P \\ F_r \end{bmatrix} \tag{22}$$

where $I_r$ is the identity matrix of order $r$ and

$$F = A_d P + A_r F_r \tag{23}$$

where

$$A_d = \begin{bmatrix} Q_d^{-1} \\ 0 \end{bmatrix}, \qquad A_r = \begin{bmatrix} -Q_d^{-1} Q_r \\ I \end{bmatrix} \tag{24}$$

Substituting Eq. (23) in the compatibility equation (15), we obtain

$$CGA_d P + CGA_r F_r = 0 \tag{25}$$

From Eq. (25), the redundant forces $F_r$ are derived from the following equations:

$$(CGA_r) F_r = -(CGA_d P) \tag{26}$$

The determinate forces $F_d$ are obtained from Eq. (21). Note that the matrices $Q$, $C$, and $G$ are banded and have full-row ranks of $m$, $r$, and $n$, respectively. The matrices $Q$ and $C$ depend on the geometry of the structure and are independent of design variables and material properties. For a finite element idealization, the generation of the equilibrium matrix $Q$ and the flexibility matrix $G$ is straightforward, and its application is found in Refs. 1–4. However, the automatic generation of the compatibility matrix $C$ is a laborious and difficult task using the SFM.

Note that the primary variables in the SFM are the redundant forces. A limitation of this method is the difficulty in automatic selection of the consistent redundant elements. Another limitation is the explicit generation of compatibility matrix $C$. In summary, the automatic generation of the compatibility matrix $C$ is cumbersome and difficult in the SFM because it is the redundant members that need to be cut and the unit-load theorem needs to be written for each cut.

### B. IFM

In the IFM, the $m$ equilibrium equations in Eq. (16) and the $r$ compatibility conditions in Eq. (15) are coupled to yield the integrated force formulation, expressed jointly as

$$\begin{bmatrix} Q \\ \cdots\cdots \\ CG \end{bmatrix} F = \begin{bmatrix} P \\ \cdots \\ 0 \end{bmatrix} \qquad \text{or} \qquad SF = P^* \tag{27}$$

where

$$S = \begin{bmatrix} Q \\ \cdots\cdots \\ CG \end{bmatrix}, \qquad P^* = \begin{bmatrix} P \\ \cdots \\ 0 \end{bmatrix}$$

The $(n \times n)$ matrix $S$ is banded and has full-row rank of $n$. In the IFM, the primary variables are the forces in the elements. Eliminating the displacements from the strain–displacement relationships without any recourse to the redundant members and the basis determinate structure generates the compatibility conditions. Following St. Venant's procedure, the compatibility conditions in the IFM are generated by eliminating the displacements from the deformation-displacement relationship of the structure. (In discrete analysis, deformations are analogous to strains.) In the IFM, the selection of redundant forces and basis determinate structure is not necessary; however, the automatic generation of compatibility equation is not straightforward.

#### 1. *Generation of the Compatibility Equations in the IFM*

The compatibility matrix $C$ is obtained by extending St. Venant's principle of elasticity strain formulation (see Ref. 18) to discrete structural mechanics.[19] This procedure is illustrated by taking the plane stress elasticity problem as an example. The strain–displacement relations are

$$\varepsilon_x \frac{\partial u_x}{\partial x}, \qquad \varepsilon_y = \frac{\partial u_y}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \tag{28}$$

In Eq. (28), three strains $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$ are expressed as functions of two displacements $u_x$ and $u_y$. The compatibility constraint on strains is obtained by eliminating the two displacements from the three relations in Eq. (28), resulting in the single compatibility condition:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \tag{29}$$

The two-step St. Venant's procedure used to generate compatibility conditions establishes the strain–displacement relations and subsequently eliminates displacements from the strain–displacement relations to obtain the compatibility conditions.

The equivalent relations of strain–displacement relations in the mechanics of discrete structures are the deformation–displacement relations. Deformations $\boldsymbol{\Delta}$ of the discrete analysis are analogous to strain $\varepsilon$ of the elasticity analysis. Thus, the deformation–displacement relations can be obtained using the following energy argument. The equality, relating internal strain energy and external work for a discrete structure $(n, m)$, can be written in the form

$$\tfrac{1}{2}\boldsymbol{F}^T\boldsymbol{\Delta} = \tfrac{1}{2}\boldsymbol{P}\boldsymbol{U} \tag{30}$$

where $\boldsymbol{U}$ is the nodal displacements vector. Equation (30) can be rewritten by eliminating the applied loads vector $\boldsymbol{P}$ in lieu of the element force vector $\boldsymbol{F}$ using the equilibrium equation (16), to obtain the following relationship:

$$\tfrac{1}{2}\boldsymbol{F}^T\boldsymbol{Q}^T\boldsymbol{U} = \tfrac{1}{2}\boldsymbol{F}^T\boldsymbol{\Delta} \qquad \text{or} \qquad \tfrac{1}{2}\boldsymbol{F}^T(\boldsymbol{Q}^T\boldsymbol{U} - \boldsymbol{\Delta}) = \boldsymbol{0} \tag{31}$$

Because the force vector $\boldsymbol{F}$ is not a null space, we obtain the following relation between member deformations and nodal displacements:

$$\boldsymbol{\Delta} = \boldsymbol{Q}^T\boldsymbol{U} \tag{32}$$

The expression given by Eq. (32) is the general deformation–displacement relation applicable to the finite element models whose equilibrium equations are given by Eq. (16). In the deformation–displacements relation, $n$ deformations are expressed in terms of $m$ displacements. Thus, there are $r = n - m$ constraints on deformations that represent the compatibility conditions of the structure $(n, m)$. The $r$ compatibility conditions, which are obtained by eliminating $m$ displacements from $n$ deformation–displacement relations, can be expressed in matrix form as

$$\boldsymbol{C}\boldsymbol{\Delta} = \boldsymbol{0} \qquad \text{or} \qquad \boldsymbol{C}\boldsymbol{G}\boldsymbol{F} = \boldsymbol{0} \tag{33}$$

Equation (33) is exactly Eq. (15). The procedure is independent of the redundant members and the basis determinate structure. The indirect generation of the $(r \times n)$ banded compatibility matrix $\boldsymbol{C}$ through selection of independent rows is amenable to computer automation and has been documented in Ref. 20.

In this study, a technique has been proposed to generate directly the compatibility matrix $\boldsymbol{C}$ in the IFM using the deformation–displacement relations (32) and the singular value decomposition (SVD) method. This enhancement to the IFM method is presented next.

*2. Alternative Method to Generate Directly the Compatibility Matrix in the IFM*

Expressing displacements in terms of deformations using Eq. (32), we obtain

$$\boldsymbol{U} = (\boldsymbol{Q}\boldsymbol{Q}^T)^{-1}\boldsymbol{Q}\boldsymbol{\Delta} = (\boldsymbol{Q}^T)^{\text{pinv}}\boldsymbol{\Delta} \tag{34}$$

The matrix $(\boldsymbol{Q}^T)^{\text{pinv}}$ is the Moore–Penrose pseudoinverse of $\boldsymbol{Q}^T$. Substituting displacements $\boldsymbol{U}$ from Eq. (34) into Eq. (32), we obtain

$$\boldsymbol{\Delta} = \boldsymbol{Q}^T(\boldsymbol{Q}^T)^{\text{pinv}}\boldsymbol{\Delta} \Rightarrow [\boldsymbol{I} - \boldsymbol{Q}^T(\boldsymbol{Q}^T)^{\text{pinv}}]\boldsymbol{\Delta} = \boldsymbol{0} \tag{35}$$

or

$$\boldsymbol{A}\boldsymbol{\Delta} = \boldsymbol{0} \tag{36}$$

where

$$\boldsymbol{A} = \left[\boldsymbol{I}_n - \boldsymbol{Q}^T(\boldsymbol{Q}^T)^{\text{pinv}}\right] \tag{37}$$

The $(n \times n)$ matrix $\boldsymbol{A}$ is of rank $r$. Note that $r < n$. This means that some of the rows of matrix $\boldsymbol{A}$ are dependent on each other. To extract the $(r \times n)$ compatibility matrix $\boldsymbol{C}$ from the matrix $\boldsymbol{A}$, that is, to reduce the matrix $\boldsymbol{A}$ to matrix $\boldsymbol{C}$, the SVD method is employed. A basic theory of the SVD method can be found in Ref. 21.

Applying SVD to $\boldsymbol{A}$, we obtain

$$\boldsymbol{A} = \boldsymbol{A}_u\boldsymbol{A}_\sigma\boldsymbol{A}_v^T \tag{38}$$

where $\boldsymbol{A}_u$ and $\boldsymbol{A}_v$ are $(n \times n)$ orthogonal matrices and

$$\boldsymbol{A}_\sigma = \begin{bmatrix} \boldsymbol{\Lambda} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}_{n \times n} \tag{39}$$

with $\boldsymbol{\Lambda} = \text{diag}\{\Lambda_1 \;\; \Lambda_2 \;\; \cdots \Lambda_r\}$ and $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_r > 0$. It follows that

$$\boldsymbol{A} = \boldsymbol{A}_u \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{0} \end{bmatrix} \tag{40}$$

Subsequently, the $(r \times n)$ compatibility matrix $\boldsymbol{C}$ can be represented as

$$\boldsymbol{C} = \boldsymbol{\Lambda}[\boldsymbol{A}_{v1} \quad \boldsymbol{A}_{v2} \quad \cdots \quad \boldsymbol{A}_{vi} \quad \cdots \quad \boldsymbol{A}_{vr}]^T \tag{41}$$

where $\boldsymbol{A}_{vi}$ is the $i$th column of matrix $\boldsymbol{A}_v$. The compatibility matrix introduced in Eq. (41) may not be banded. This does not raise any problem for relatively small-size problems; however, it may be numerically expensive for very large-size problems.

The direct generation of the compatibility matrix proposed and developed here to enhance the IFM method has been implemented successfully for analysis of truss- and beam-type structures.

*3. Direct Displacement–Force Relations*

Although Eq. (32) is sufficient to obtain the element deformations using nodal displacements, it is not sufficient to obtain nodal displacements using element deformations or forces because redundant structures are represented by rectangular equilibrium matrix $\boldsymbol{Q}$ with no inverse. This implies that the compatibility equations should be merged with the equilibrium equations. For this reason, using $\boldsymbol{S}$ instead of $\boldsymbol{Q}$ in Eq. (32) and solving for nodal displacements $\boldsymbol{U}$, we obtain

$$\boldsymbol{U} = \boldsymbol{J}\boldsymbol{\Delta} \qquad \text{or} \qquad \boldsymbol{U} = \boldsymbol{J}\boldsymbol{G}\boldsymbol{F} \tag{42}$$

where

$$\boldsymbol{J} = m \text{ rows of } \boldsymbol{S}^{-T} \tag{43}$$

### C. Force Method Based on the Complementary Strain Energy

The main contribution of this study has been to develop a new force methodology based on the complementary strain energy (FMCE). The $m$ equilibrium equations are satisfied, and subsequently the complementary strain energy is minimized with respect to the $r$ redundant forces to satisfy the $r$ compatibility equations. Similar to the SFM, it is required to identify the redundant elements and the basis determinate elements. However, the generation of the compatibility matrix $\boldsymbol{C}$ is circumvented, making the proposed method more amenable to automation. Consider Eq. (20),

$$\boldsymbol{Q}_r\boldsymbol{F}_r + \boldsymbol{Q}_d\boldsymbol{F}_d = \boldsymbol{P} \tag{44}$$

solving for $\boldsymbol{F}_d$, we obtain

$$\boldsymbol{F}_d = \boldsymbol{H}_1 - \boldsymbol{H}_2\boldsymbol{F}_r \tag{45}$$

where

$$\boldsymbol{H}_1 = \boldsymbol{Q}_d^{-1}\boldsymbol{P}, \qquad \boldsymbol{H}_2 = \boldsymbol{Q}_d^{-1}\boldsymbol{Q}_r \tag{46}$$

The flexibility matrix $\boldsymbol{G}$ may be consistently partitioned for $\boldsymbol{F}_d$ and $\boldsymbol{F}_r$ as

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{G}_d & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_r \end{bmatrix} \tag{47}$$

where $\boldsymbol{G}_d$ and $\boldsymbol{G}_r$ are $(m \times m)$ and $(r \times r)$ flexibility matrices related to the basis determinate structure and the redundant elements. Thus, the complementary strain energy can be expressed as

$$\Pi^* = \tfrac{1}{2}\boldsymbol{F}^T\boldsymbol{\Delta} \qquad \text{or} \qquad \Pi^* = \tfrac{1}{2}\boldsymbol{F}^T\boldsymbol{G}\boldsymbol{F} \tag{48}$$

Substituting Eq. (18) and Eq. (47) into Eq. (48), we obtain

$$\Pi^* = \tfrac{1}{2}\boldsymbol{F}_d^T\boldsymbol{G}_d\boldsymbol{F}_d + \tfrac{1}{2}\boldsymbol{F}_r^T\boldsymbol{G}_r\boldsymbol{F}_r \tag{49}$$

Substituting Eq. (45) into Eq. (49), we obtain

$$\Pi^* = \tfrac{1}{2}(\boldsymbol{H}_1 - \boldsymbol{H}_2\boldsymbol{F}_r)^T\boldsymbol{G}_d(\boldsymbol{H}_1 - \boldsymbol{H}_2\boldsymbol{F}_r) + \tfrac{1}{2}\boldsymbol{F}_r^T\boldsymbol{G}_r\boldsymbol{F}_r \Rightarrow$$

$$\Pi^* = \tfrac{1}{2}\boldsymbol{F}_r^T\boldsymbol{G}_r\boldsymbol{F}_r + \tfrac{1}{2}\boldsymbol{F}_r^T\left(\boldsymbol{H}_2^T\boldsymbol{G}_d\boldsymbol{H}_2\right)\boldsymbol{F}_r - \tfrac{1}{2}\boldsymbol{F}_r^T\left(\boldsymbol{H}_2^T\boldsymbol{G}_d\boldsymbol{H}_1\right)$$
$$-\tfrac{1}{2}\left(\boldsymbol{H}_1^T\boldsymbol{G}_d\boldsymbol{H}_2\right)\boldsymbol{F}_r + \tfrac{1}{2}\boldsymbol{H}_1^T\boldsymbol{G}_d\boldsymbol{H}_1 \tag{50}$$

The matrix $\boldsymbol{G}_d$ is symmetric, $\boldsymbol{F}_r^T(\boldsymbol{H}_2^T\boldsymbol{G}_d\boldsymbol{H}_1) = (\boldsymbol{H}_1^T\boldsymbol{G}_d\boldsymbol{H}_2)\boldsymbol{F}_r$, and thus Eq. (50) can be expressed as

$$\Pi^* = \tfrac{1}{2}\boldsymbol{F}_r^T\boldsymbol{G}_r\boldsymbol{F}_r + \tfrac{1}{2}\boldsymbol{F}_r^T\left(\boldsymbol{H}_2^T\boldsymbol{G}_d\boldsymbol{H}_2\right)\boldsymbol{F}_r - \boldsymbol{F}_r^T\left(\boldsymbol{H}_2^T\boldsymbol{G}_d\boldsymbol{H}_1\right) + \tfrac{1}{2}\boldsymbol{H}_1^T\boldsymbol{G}_d\boldsymbol{H}_1 \tag{51}$$

Minimization of Eq. (51) with respect to redundant forces $\boldsymbol{F}_r$ gives the $r$ compatibility equations:

$$\frac{\partial\Pi^*}{\partial\boldsymbol{F}_r} = 0 \Rightarrow \left(\boldsymbol{H}_2^T\boldsymbol{G}_d\boldsymbol{H}_2 + \boldsymbol{G}_r\right)\boldsymbol{F}_r = \boldsymbol{H}_2^T\boldsymbol{G}_d\boldsymbol{H}_1 \tag{52}$$

Here, the $r$ redundant forces are obtained and subsequently the $m$ basis determinate forces are determined from Eq. (45). Note that the proposed technique presents a marked advantage over the other force methods because it does not require the explicit generation of the compatibility matrix $\boldsymbol{C}$.

The selection of the redundant members is not unique, and there are multiple combinations of matrices $\boldsymbol{Q}_d$ and $\boldsymbol{Q}_r$ for an indeterminate structure. For example, a simple structure with $m = 5$ and $n = 20$ can have a maximum of 15,504 probable combinations of $\boldsymbol{Q}_d$ and $\boldsymbol{Q}_r$. Redundant forces should be selected so that the remaining determinate structure is not a mechanism. In other words, the selection of the consistent set of redundant members and basis determinate structure is such that the rank of the matrix $\boldsymbol{Q}_d$ is equal to $m$. The violation of this condition makes the matrix $\boldsymbol{Q}_d$ singular. Here, a robust technique based on the Gauss elimination technique is applied to identify automatically the consistent set of redundant members and basis determinate structure. The technique was introduced by Robinson[11] for identifying dependent and independent equations among a system of static equations including external, joint, and element equilibrium equations. Here this technique has been modified and applied to redundant structures to identify the consistent set of redundant members and basis determinate structure. The proposed technique is outlined as follows:

1) Augment the equilibrium matrix $Q$ with the external load $P$ as $[Q \;\; P]$.

2) Select one of the nonzero elements in the first row of the augmented matrix, and divide all elements in this row by this number.

3) Multiply the first row by the coefficient of the corresponding element in the second row (if it is not zero) and subtract from the second row.

4) Continue this procedure for each of the remaining rows.

5) The column corresponding to that element has now a one in the first row and zeros in all other rows.

6) Repeat the same process from steps 2–4, in turn for the remaining rows until either all of the rows are exhausted or all of the remaining rows have all zeros as elements.

7) All of the $m$ unit columns are independent, and they correspond to the basis determinate structure. The remaining columns correspond to the consistent redundant members.

8) The consistent redundant members selected are not unique because the redundancy is dependent on the order in which the equations are generated and by the selection of the nonzero element in each row when applying the Gaussian elimination procedure. This point is illustrated next.

To illustrate the concept, consider a 10-bar truss structure shown in Fig. 1. This structure has $r = n - m = 10 - 8 = 2$ redundant members. It is required to identify the redundant members and to eliminate them so that the basis determinate structure is stable. (The rank of the basis matrix is $m$.) The equilibrium matrix $Q$ for this structure is

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \sqrt{2}/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\sqrt{2}/2 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & \sqrt{2}/2 & 0 & 0 & -\sqrt{2}/2 \\ 0 & 0 & 0 & 0 & -1 & 0 & -\sqrt{2}/2 & 0 & 0 & -\sqrt{2}/2 \end{bmatrix} \tag{53}$$

Now, select the maximum nonzero element in each row. At the end of the Gauss elimination procedure, the matrix $Q$ is transformed into

$$\boldsymbol{Q} = \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{54}$$

Here, columns 1 and 2 corresponding to elements 1 and 2, respectively, represent the consistent redundant members. Columns 3–10 corresponding to elements 3–10 represent the consistent determinate basis structure. Because columns 3–10 are independent, the rank of the determinate basis structure is $m = 8$. Now, select the first nonzero element in each row. At the end of the Gauss elimination, the matrix $Q$ is transformed into

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \sqrt{2}/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \tag{55}$$

Here, columns 8 and 10 corresponding to elements 8 and 10, respectively, represent the consistent redundant members. Columns 1, 2, 3, 4, 5, 6, 7, and 9 corresponding to the elements 1, 2, 3, 4, 5, 6, 7, and 9 represent the consistent determinate basis structure. Observe that columns 1, 2, 3, 4, 5, 6, 7, and 9 are independent. Thus, the rank of the determinate basis structure is $m = 8$ again.

In summary, it is possible to identify the consistent redundant elements and basis determinate structure using the Gauss elimination procedure. This method is easily implemented computationally. The consistent redundant members are not unique and depend on which nonzero element is selected in each row.

**Table 1 Advantages and limitations of FMs**

| Advantages | Limitations |
|---|---|
| | *SFM* |
| Need $r$ equations to be solved instead of the $m$ equations in the displacement method. | Primary variables are redundant members ⇒ consistent set of redundant members need to be selected. Generation of compatibility matrix needs cutting the redundant members ⇒ difficult to automate. |
| | *IFM* |
| No need to select redundant members. Compatibility matrix can be obtained automatically and directly. | Generation of compatibility matrix. Need $n$ equations to be solved simultaneously in comparison to the $m$ equations in the displacement method. |
| | *FMCE* |
| No need to generate compatibility matrix. | Selection of the consistent redundant members and determinate basis structure. |
| Can be automated easily by Gaussian elimination procedure. Need $r$ equations to be solved simultaneously. | |

### D. Comparison of the Force Methods

The merits and limitations of the force methods presented and discussed are identified in Table 1.

## III. Comparison Between the Force and Displacement Methods

In the displacement method, the compatibility equations are implicitly satisfied when the nodal equilibrium equations are written. The governing equation in the displacement method (load–nodal displacement relations) may be derived using the force method given by Eqs. (8), (16), and (32) promptly, but the reverse is not true.[18] Substituting deformation $\boldsymbol{\Delta}$ from Eq. (32) into Eq. (8) and solving for the element forces, we obtain

$$\boldsymbol{F} = \boldsymbol{G}^{-1}\boldsymbol{Q}^T\boldsymbol{U} \tag{56}$$

Now substitute Eq. (56) into the equilibrium equation (16):

$$(\boldsymbol{Q}\boldsymbol{G}^{-1}\boldsymbol{Q}^T)\boldsymbol{U} = \boldsymbol{P} \Rightarrow \boldsymbol{K}\boldsymbol{U} = \boldsymbol{P} \tag{57}$$

where

$$\boldsymbol{K} = \boldsymbol{Q}\boldsymbol{G}^{-1}\boldsymbol{Q}^T \tag{58}$$

Here, the matrix $\boldsymbol{K}$ is the stiffness matrix used in the displacement method. Note that because the equilibrium and the compatibility equations are satisfied explicitly in the force method, it is not possible to derive the governing equation in the force method [Eq. (16)] from Eq. (57), confirming the noncommutative property of the two formulations.

In the force method, the element forces $\boldsymbol{F}$ are calculated directly from the loads. On the other hand, in the displacement method, one has to calculate the nodal displacements $\boldsymbol{U}$ from the loads using the load–displacement relation (57). In practical design and optimization problems, it is required to obtain element forces in all elements, and fewer nodal displacements may be necessary. To illustrate this, consider structural optimization problems, in general. Here, the number of stress constraints is usually greater than the number of displacement constraints. The computation time is considerably reduced using force method. This is attributed to the following factors.

The displacement method requires a series of transformations and back substitutions (from local to global system to generate displacements and then from global to local system to calculate the forces). In the force methods, these transformations are not required.

In the displacement method, $m$ equations are solved simultaneously, whereas only $r$ equations are solved using the force method.

In the force method, the equilibrium matrix $\boldsymbol{Q}$ for the general determinate structure, after some rearrangement of rows and columns, can be represented as a triangular matrix. The stiffness matrix in the displacement method is not a triangular matrix for the determinate structure. The triangular system of equations requires insignificant computation, and it can be solved even manually irrespective of the size or complexity of the problem. This feature of the force method made it the popular analysis method in the precomputer era.

In the force method, the coefficients of the equilibrium matrix $Q$ and the compatibility matrix are dimensionless numbers, which makes them numerically stable. On the other hand, the coefficients of the stiffness matrix in the displacement method have the dimensions of force per unit length. Because the coefficients depend on material properties and design parameters, an ill-conditioned stiffness matrix can result with a numerically unfavorable combination of these properties.[18]

## IV. Extension of the Force Method to Dynamics

The force method can be extended to analyze dynamic problems.[22,23] In the displacement method, the basic equation in dynamics problems in the absence of damping is

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{P} \tag{59}$$

where $\boldsymbol{M}$ is the mass matrix of system. When Eq. (42) is used and it is noted that $\boldsymbol{KU}$ in the displacement method is equivalent to $\boldsymbol{SF}$ in the force method, Eq. (59) may be written as

$$\boldsymbol{M}^*\ddot{\boldsymbol{F}} + \boldsymbol{S}\boldsymbol{F} = \boldsymbol{P}^* \tag{60}$$

where

$$\boldsymbol{M}^* = \begin{bmatrix} \boldsymbol{MJG} \\ \cdots \\ \boldsymbol{0} \end{bmatrix} \tag{61}$$

Equation (60) represents the dynamic problems in the framework of the force formulation. In forced vibration problems, $\boldsymbol{P}^*$ is a function of time. There are a few time-integration techniques in the literature for solving Eq. (60). One of the most powerful techniques is the Newmark direct integration method,[24] which is based on the modified average acceleration method. It is assumed that the initial values of force vector $\boldsymbol{F}$ and the vector $\dot{\boldsymbol{F}}$ at time $t = 0$ is known. The vector $\ddot{\boldsymbol{F}}$ at $t = 0$ is obtained from Eq. (60). When a time increment $\Delta t$ is considered, the predictor parameters $\widehat{\boldsymbol{F}}_{n+1}$ and $\hat{\dot{\boldsymbol{F}}}_{n+1}$ at time $(n+1)\Delta t$ in terms of the known vectors at time $n\Delta t$ are computed as

$$\widehat{\boldsymbol{F}}_{n+1} = \boldsymbol{F}_n + \Delta t\dot{\boldsymbol{F}}_n + 0.5\Delta t^2(1-2\beta)\ddot{\boldsymbol{F}}_n$$
$$\hat{\dot{\boldsymbol{F}}}_{n+1} = \dot{\boldsymbol{F}}_n + \Delta t(1-\gamma)\ddot{\boldsymbol{F}}_n \tag{62}$$

Now, the vector $\ddot{\boldsymbol{F}}$ at time $(n+1)\Delta t$, that is, $\ddot{\boldsymbol{F}}_{n+1}$, is obtained from the following equation:

$$(\boldsymbol{M}^* + \beta\Delta t^2\boldsymbol{S})\ddot{\boldsymbol{F}}_{n+1} = \boldsymbol{P}^*_{n+1} - \boldsymbol{S}\widehat{\boldsymbol{F}}_{n+1} \tag{63}$$

When $\ddot{F}_{n+1}$ is known, the force vector $F$ and the vector $\dot{F}$ at time $(n+1)\Delta t$ are obtained from the following relations:

$$F_{n+1} = \hat{F}_{n+1} + \beta \Delta t^2 \ddot{F}_{n+1}, \qquad \dot{F}_{n+1} = \dot{\hat{F}}_{n+1} + \gamma \Delta t \ddot{F}_{n+1} \quad (64)$$

Constants $\beta$ and $\gamma$ in the preceding equations are the accuracy and stability parameters in the Newmark method. The Newmark method is unconditionally stable if $0.5 \leq \gamma \leq 2\beta$. Note that Eqs. (62–64) are Newmark formulations in the force format.

In free vibration, it is assumed that element forces are harmonics in time $[F = \bar{F}\sin(\omega t)]$, where $\omega$ and $\bar{F}$ are frequency and force mode shape, respectively. Consider Eq. (60),

$$S\bar{F} - \omega^2 M^* \bar{F} = 0 \quad (65)$$

To overcome some computational difficulties during the analysis, the $(n \times n)$ system of Eqs. (65) can be reduced to an $(m \times m)$ system by taking advantage of the null matrices. To obtain these matrices, the matrices in Eq. (65) are partitioned as the redundant and basis determinate structure as

$$\begin{bmatrix} S_{dd} & S_{dr} \\ S_{rd} & S_{rr} \end{bmatrix} \begin{bmatrix} \bar{F}_d \\ \bar{F}_r \end{bmatrix} - \omega^2 \begin{bmatrix} \tilde{M}_r & \tilde{M}_d \\ \cdots\cdots & \\ \mathbf{0} & \end{bmatrix} \begin{bmatrix} \bar{F}_d \\ \bar{F}_r \end{bmatrix} = \mathbf{0} \quad (66)$$

$$S_{dd}\bar{F}_d + S_{dr}\bar{F}_r - \omega^2(\tilde{M}_d\bar{F}_d + \tilde{M}_r\bar{F}_r) = 0 \quad (67a)$$

or

$$S_{rd}\bar{F}_d + S_{rr}\bar{F}_r = 0 \quad (67b)$$

Eliminating $F_r$ from the $(n \times n)$ system of Eqs. (67) results in the reduced $(m \times m)$ subsystem:

$$\left(S_{dd} - S_{dr}S_{rr}^{-1}S_{rd}\right)\bar{F}_d - \omega^2\left(\tilde{M}_d - \tilde{M}_r S_{rr}^{-1}S_{rd}\right)\bar{F}_d = \mathbf{0} \quad (68)$$

$$\bar{F}_r = S_{rr}^{-1}S_{rd}\bar{F}_d \quad (69)$$

Selection of consistent redundant members ensures the existence of the inverse of $S_{22}$.

The solution of the reduced eigenvalue problem expressed by Eq. (68) gives all of the eigenvalues, whereas both Eqs. (68) and (69) are used to calculate the force eigenvectors. Once the force mode shapes are known, the displacement mode shapes can be generated by using Eq. (42).

### A. Ten-Bar Planar Truss

In this example, the weight of a 10-bar planar truss shown in the Fig. 1 has been been minimized under multiple frequency constraints. The objective is to show the efficiency and accuracy of the force method in comparison to the displacement method in structural optimization problems. The element cross-sectional areas are considered as design variables. The material is aluminum with Young's modulus $E = 10^7$ psi ($6.89 \times 10^{10}$ Pa) and density $\rho = 0.1$ lbm/in.$^3$ (2770 kg/m$^3$). The minimum area for all elements was set at 0.1 in.$^2$ (0.645 cm$^2$) and $L = 360$ in (914.4 cm). At each of the four free nodes, a nonstructural lumped mass of 1000 lbm (2.588 lb · s$^2$/in.) (454 kg) is added.

The number of DOF is $m = 8$, and the number of FOF is $n = 10$. Thus, the number of redundancy is $r = 2$. At the initial design, all of the cross-sectional areas are 20 in.$^2$ (129.03 cm$^2$), and the initial mass is 8392.94 lbm (3810.39 kg). Here the force method has been used for analysis and sequential programming technique (SQP) for optimization. The implementation of the SQP has been done in MATLAB®.[25]

This problem was investigated by Venkayya and Tischler,[26] as well as by Grandhi and Venkayya,[27] using the optimality criterion and displacement method. First, the structure was designed with a fundamental frequency of 14 Hz alone, using both the displacement and force methods. A minimum weight of 5810.24 lbm (2637.85 kg) was obtained, and the first natural frequency in the optimum design was found to be exactly 14 Hz. The number of iterations required using the force method (FM) was lower than that required by the displacement method (DM). It was observed that the computational time for the FM was about two times less than that of DM. The final results for the cross-sectional areas and fundamental frequency are tabulated in Tables 2 and 3, respectively. Venkayya and Tischler[26] have reported a minimum mass of 6665.577 lbm (3026.17 kg). The optimum design in Ref. 26 was taken as input to compute the specified natural frequency. A fundamental natural frequency of 14.47 Hz was obtained in the present analysis. Another simulation carried out using the solution reported in Ref. 26 as the initial design resulted in a final design, which again converged to a lighter solution 5810.24 lbm (2637.85 kg) obtained earlier.

To demonstrate the application of the formulation for designing a structure with other specified natural frequencies, the structure was designed for a second natural frequency of 25 Hz. A minimum mass of 1920.52 lbm (871.92 kg) was obtained, and the second natural frequency in optimum design was found to be exactly 25 Hz. The computational time using the FM was about two times less than that of the DM. Grandhi and Venkayya[27] reported a minimum mass of 2243.8 lbm (1018.69 kg). Here, the optimum design in Ref. 27 was used as input to compute the second natural frequency, resulting in a solution of 25.37 Hz for the second natural frequency. The final results for the cross-sectional areas in square centimeters and the fundamental frequency in hertz are given in Tables 2 and 3, along with the number of active constraints (AC).

**Table 2 Final design for the cross-sectional areas for various frequency constraints**

| | DM, cm$^2$ | | | FM, cm$^2$ | | |
|---|---|---|---|---|---|---|
| Element | $\omega_1 = 14$ | $\omega_2 = 25$ | $\omega_1 = 7$, $\omega_2 \geq 15$, $\omega_3 \geq 20$ | $\omega_1 = 14$ | $\omega_2 = 25$ | $\omega_1 = 7$, $\omega_2 \geq 15$, $\omega_3 \geq 20$ |
| 1 | 219.909 | 48.166 | 38.619 | 219.903 | 48.123 | 38.245 |
| 2 | 47.916 | 35.852 | 18.239 | 47.916 | 35.832 | 9.916 |
| 3 | 219.909 | 48.194 | 38.252 | 219.903 | 48.200 | 38.619 |
| 4 | 47.916 | 35.852 | 9.910 | 47.916 | 35.884 | 18.232 |
| 5 | 0.645 | 14.800 | 4.419 | 0.645 | 14.826 | 4.419 |
| 6 | 0.645 | 7.632 | 4.200 | 0.645 | 7.632 | 4.194 |
| 7 | 123.626 | 41.135 | 24.110 | 123.626 | 41.103 | 20.097 |
| 8 | 123.626 | 41.142 | 20.084 | 123.626 | 41.181 | 24.097 |
| 9 | 54.677 | 13.200 | 11.452 | 54.677 | 13.200 | 13.890 |
| 10 | 54.677 | 13.194 | 13.897 | 54.677 | 13.187 | 11.4516 |
| Mass, kg | 2637.85 | 871.92 | 537.01 | 2637.85 | 871.92 | 537.01 |
| Number of AC | 3 | 1 | 2 | 3 | 1 | 2 |

**Table 3 Final design of natural frequencies in different frequency constraints**

| Frequency number | Initial design, Hz | DM, Hz $\omega_1 = 14$ | DM, Hz $\omega_2 = 25$ | DM, Hz $\omega_1 = 7$, $\omega_2 \geq 15$, $\omega_3 \geq 20$ | FM, Hz $\omega_1 = 14$ | FM, Hz $\omega_2 = 25$ | FM, Hz $\omega_1 = 7$, $\omega_2 \geq 15$, $\omega_3 \geq 20$ |
|---|---|---|---|---|---|---|---|
| 1 | 11.23 | 14.00 | 8.01 | 7.00 | 14.00 | 8.01 | 7.00 |
| 2 | 33.05 | 18.01 | 25.00 | 17.62 | 18.01 | 25.00 | 17.62 |
| 3 | 36.85 | 29.40 | 25.00 | 20.00 | 29.40 | 25.00 | 20.00 |
| 4 | 68.26 | 34.55 | 26.68 | 20.00 | 34.55 | 26.68 | 20.00 |
| 5 | 75.86 | 49.36 | 32.83 | 28.20 | 49.36 | 32.83 | 28.21 |
| 6 | 85.18 | 53.11 | 40.92 | 31.07 | 53.11 | 40.94 | 31.07 |
| 7 | 85.74 | 85.10 | 62.52 | 47.68 | 85.10 | 62.52 | 47.68 |
| 8 | 103.10 | 90.41 | 64.79 | 52.35 | 90.41 | 64.78 | 52.35 |

**Table 4 Final design of natural frequencies in different frequency constraints for the member frame structure**

| Frequency number | Initial design, rad/s | DM, rad/s $\omega_1 = 78.5$ | DM, rad/s $\omega_1 = 78.5$, $\omega_2 \geq 180$ | FM, rad/s $\omega_1 = 4$ | FM, rad/s $\omega_1 = 78.5$, $\omega_2 \geq 180$ |
|---|---|---|---|---|---|
| 1 | 69.044 | 78.500 | 78.500 | 78.500 | 78.500 |
| 2 | 286.840 | 146.670 | 2220.806 | 146.668 | 180.000 |
| 3 | 380.324 | 268.399 | 436.420 | 268.350 | 371.289 |
| 4 | 476.168 | 350.723 | 486.975 | 350.667 | 418.804 |
| 5 | 499.720 | 465.900 | 540.125 | 465.780 | 485.897 |

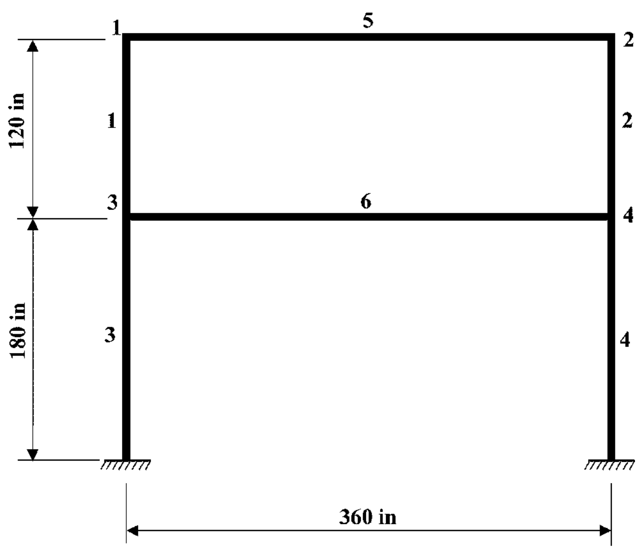


**Fig. 2 Six-member frame structure.**

Finally, the structure was designed under multiple natural frequency constraints given by $\omega_1 = 7$ Hz, $\omega_2 \geq 15$ Hz, and $\omega_3 \geq 20$ Hz. A minimum mass of 1182.85 lbm (537.01 kg) was obtained. On closer inspection, the results reveal that the optimum cross-sectional areas for elements 9 and 10 obtained using the FM are different from those using the DM. Note that the optimum masses for both the FM and DM are exactly the same and so are the final natural frequencies. It is inferred that, because the optimizer is very sensitive to the output results from the FM and the DM, a small difference causes the optimizer to select a different path. Again computational time using the FM is two times less than that of the DM. Grandhi and Venkayya[27] have reported a minimum weight of 1308.4 lbm (594.01 kg). The final cross-sectional areas and natural frequencies are given in the Tables 2 and 3.

### B. Six-Member Frame

The six-member frame is shown in Fig. 2. This problem has been studied by Khan and Willmert[28] and MacGee and Phan[29] using the optimality criterion method as an optimizer and the finite element based on the displacement method as an analyzer. A uniformly distributed nonstructural mass of 10 lbm/in. (1.79 kg/cm) was added on the horizontal members of the frame. The density and Young's modulus are 0.28 lbm/in.$^3$ ($7.76 \times 10^{-3}$ kg/cm$^3$) and 30,000,000 psi (20.67 Pa), respectively. Each member is a WF steel section, according to the American Institute of Steel Constructions (AISC) code. The moment of inertia $I$ is empirically related to area $A$ by the following expressions:

$$I = 4.6248A^2, \qquad 0 \leq A \leq 44$$

$$I = 256A - 2300, \qquad 44 < A \leq 88.2813$$

where $A$ has a dimension in square inches. A minimum constraint on the design variables (cross-sectional area of the members) was specified at 7.9187 in.$^2$ (51.09 cm$^2$), and a maximum was set at 88.28 in.$^2$ (569.55 cm$^2$). At the initial design stage, all of the cross-sectional areas are equal to 30 in.$^2$ (193.55 cm$^2$) with an initial mass of 11,088 lbm (5033.95 kg)

First, the structure was designed for a fundamental natural frequency of 78.5 rad/s. A minimum mass of 9410.39 lbm (4272.32 kg) was obtained using both the FM and the DM. Note that the final design variables (cross-sectional areas) are different between the DM and FM solutions, which points out that the optimum solution is not unique. However, the final natural frequencies are the same. The reason for this difference between the optimum results using the DM and the FM is that the final design depends on the path taken over the process of optimization. Depending on the problem, the optimizer may be very sensitive to the output results from the analyzer so that even a slight difference may cause a difference path. Note that although the optimum results obtained through the FM and DM are different, they resulted to the same optimum mass and the same final natural frequencies. Therefore, both are optimum solutions. The number of iterations and CPU time in the DM are slightly greater than those of the FM. The results are given in the Tables 4 and 5. In Refs. 28 and 29 minimum weights of 9561 lbm (4340.69 kg) and 9815 lbm (4456.01 kg) are reported, respectively. To check for the possibility that the optimality criterion employed produced a local minimum, another run was performed that started with the solution from Refs. 28 and 29. This run resulted in a design change and converged to the lighter solution of 9410.39 lbm (4272.32 kg) obtained earlier.

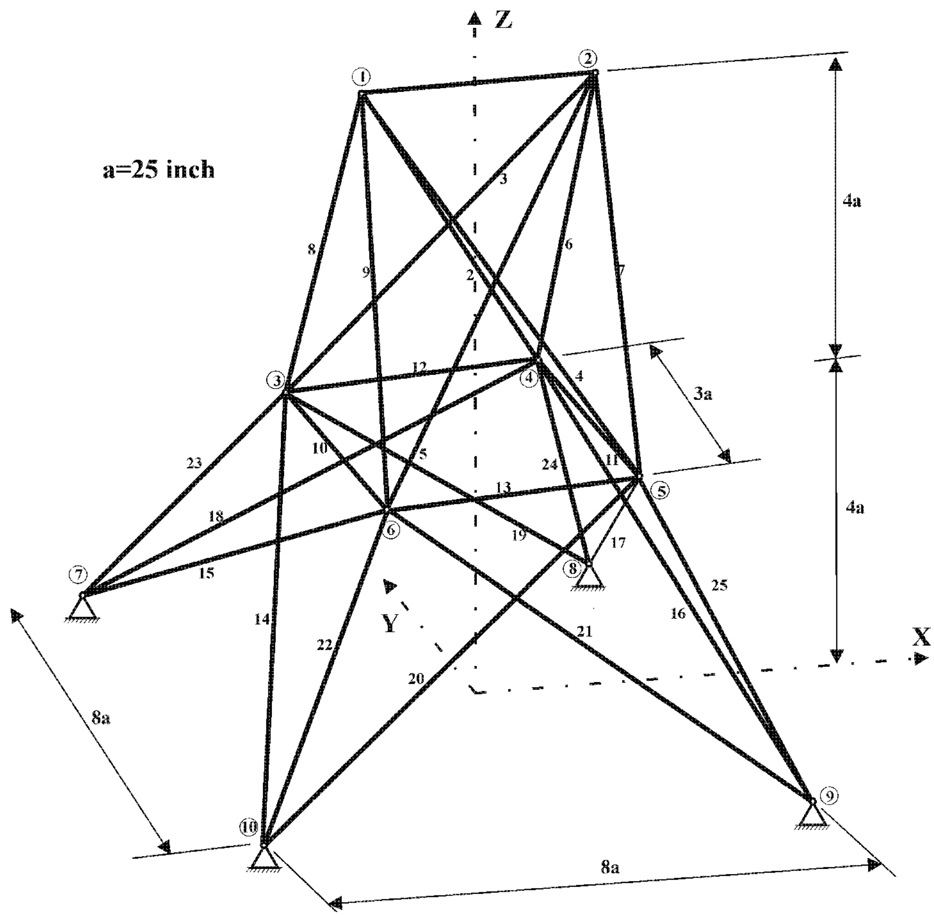


**Fig. 3 Steel 25-bar space truss structure.**

**Table 5 Final design for the cross-sectional areas ($mm^2$) for different frequency constraints (rad/s) for the six-member frame structure**

| | DM | | FM | |
|---|---|---|---|---|
| Element | $\omega_1 = 78.5$ | $\omega_1 = 78.5$ $\omega_2 \geq 180$ | $\omega_1 = 78.5$ | $\omega_1 = 78.5$ $\omega_2 \geq 180$ |
| 1 | 21,586.67 | 12,055.59 | 5,108.83 | 20,628.93 |
| 2 | 5,108.83 | 14,180.17 | 21,555.12 | 6,274.57 |
| 3 | 5,108.83 | 28,387.04 | 36,598.18 | 13,876.68 |
| 4 | 36,720.31 | 22,776.08 | 5,108.83 | 29,787.55 |
| 5 | 5,108.83 | 5,108.83 | 5,108.83 | 5,108.83 |
| 6 | 25,308.72 | 22,854.92 | 25,379.88 | 25,636.08 |
| Mass, kg | 4,272.32 | 4,418.46 | 4,272.31 | 4,365.56 |
| No. of iterations | 320 | 726 | 258 | 246 |
| No. of AC | 4 | 2 | 4 | 3 |
| CPU time, s | 15.31 | 34.29 | 11.76 | 11.21 |

The structure was again designed using multiple natural frequencies of $\omega_1 = 78.5$ and $\omega_2 \geq 180$. Surprisingly, optimum mass of 9615.78 lbm (4365.56 kg) using the FM and 9732.3 lbm (4418.46 kg) using the DM was obtained. As explained before, this specific problem is path dependent and slight differences in output results from analyzers (FM and DM) may cause different optimum solution. Investigating the final natural frequencies for DM and FM reveals that in FM the inequality constraint is active in the optimum solution, but this is not the case for DM, and that is the reason for the lighter mass obtained using FM. For this case, note that the number of iteration and CPU time in FM is lower than that of DM.

### C. Twenty-five-Bar Space Structure

The weight of 25-bar space truss shown in Fig. 3 is minimized under displacement constraints. This problem has been investigated by Saka[30] using the DM and optimality criterion approach.

**Table 6 Nodal load components for 25-bar space truss structure**

| | Coordinate directions | | |
|---|---|---|---|
| Node | $X$, N | $Y$, N | $Z$, N |
| 1 | 80,000 | 120,000 | 30,000 |
| 2 | 60,000 | 100,000 | 30,000 |
| 3 | 30,000 | 0 | 0 |
| 6 | 30,000 | 0 | 0 |

The material is steel with $E = 207$ kN/mm$^2$ and $\rho = 7830$ kg/m$^3$. The structure has identical symmetries about the $X$–$Z$ and $Y$–$Z$ planes, and so design variable linking is used to impose symmetry on the structure. Thus, only eight design variables are required. The structure is subjected to one load case, shown in Table 6. The tension allowable stress for all elements is $\sigma_{at} = 240$ N/mm$^2$; however, allowable compression stress is obtained according to AISC,[31] which is as follows. For $S_R > C_c$, $\sigma_{ac} = \pi^2 E / S_R^2$, and for $S_R < C_c$, $\sigma_{ac} = \sigma_{at}(1 - 0.5 S_R^2 / C_c^2)$. $S_R$ is the slender ratio of each member ($S_R = L/R_G$, where $L$ is the length and $R_G$ is the radius of gyration for each member) and $C_c = \sqrt{(2\pi^2 E/\sigma_{at})}$. Therefore, the allowable compression stress varies during the optimization process. Stress constraints have been imposed on all elements. Displacement constraints of $\pm 10$ mm are imposed on nodes 1 and 2 in the $X$ and $Y$ directions. The minimum area for all elements was set at 200 mm$^2$. The members have pipe-type sections with $S_R = aA^b$, where $a$ and $b$ are 0.4993 and 0.6777, respectively. $A$ is the area in square centimeters. The number of DOF is $m = 18$, and the number of FOF is $n = 25$. Therefore, the number of redundancy is $r = 7$. Without linking the design variables, the number of design variables is 25,

**Table 7 Final design of cross-sectional areas for 25-bar space structure**

| Design variable | Member | DM | FM |
|---|---|---|---|
| 1, $mm^2$ | 1 | 232.7 | 232.7 |
| 2, $mm^2$ | 2–5 | 1150.6 | 1150.6 |
| 3, $mm^2$ | 6–9 | 895.1 | 895.1 |
| 4, $mm^2$ | 10, 11 | 230.4 | 230.4 |
| 5, $mm^2$ | 12, 13 | 223.3 | 223.3 |
| 6, $mm^2$ | 14–17 | 1018.4 | 1018.4 |
| 7, $mm^2$ | 18–21 | 950.2 | 950.2 |
| 8, $mm^2$ | 22–25 | 1443.5 | 1443.5 |
| Mass, kg | | 649.7 | 649.7 |
| AC | | 8 | 8 |

and the number of the constraints is 56. When the design variables are linked into 8 groups, the number of design variables becomes 8, and the number of the constraints reduces to 22.

A minimum mass of 649.7 kg was obtained using both the DM and FM. The final results is given in Table 7. The initial cross-sectional area for all of the elements is 2000 $mm^2$. The CPU time required for the FM is significantly lower than the DM (about four times), which points out the efficiency of the FM. The compression stress constraints in elements 1, 2, 6, 10, 13, 16, 18, and 24 (one member from each group) are active. The problem was solved using different initial areas for all elements. The results were exactly the same.

A minimum weight of 921 kg was obtained by Saka[30] using DM and optimality criterion approach. He used the optimality criterion based only on satisfying the displacement constraints. The stress constraints were satisfied through the stress-ratio technique. Fleury and Schmit[32] also solved the problem using the dual methods and approximation concepts considering DM. This structure was also analyzed using the data provided by Fleury and Schmit, and identical results were obtained.

## V. Conclusions

The FMCE is proposed, and its merits and limitations are compared to the classical FM and the more recent IFM. In the proposed FM, the compatibility conditions are satisfied through the complementary energy. Automatic generation of a basis structure and redundant members using the Gauss elimination technique has been successfully demonstrated for truss- and beam-type structures. In the IFM, an efficient method based on the SVD technique has been developed to generate the compatibility conditions directly. The application of the FM to structures with small redundancies has proved to be computationally more efficient than the DM. Extension of the FM to dynamics is presented and applied to structural design optimization problems under displacement–stress and frequency constraints.

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